ON A CLASS OF NONLINEAR SECOND-ORDER DIFFERENTIAL EQUATIONS(1)

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1. In this paper we shall be concerned with differential equations of the form

$$y'' + yF(y^2, x) = 0, \tag{1}$$

where the function $F(t, x)$ is subject to the following conditions:

(2a) $F(t, x)$ is continuous in both t and x for $0 \leqq t < \infty$ and $0 < x < \infty$;

(2b) $F(t, x) > 0$ for $t > 0$, $x > 0$;

(2c) $t_2^{-\epsilon} F(t_2, x) > t_1^{-\epsilon} F(t_1, x)$ for $0 \leqq t_1 < t_2 < \infty$, fixed positive x, and some positive number ϵ.

Because of (2b) we have $yy'' < 0$ for $y \neq 0$, i.e., all solution curves of (1) are concave toward the horizontal axis. Accordingly in an interval in which $y(x)$ does not change its sign, the curve $y = y(x)$ must lie between the x-axis and the tangent to the curve at any point of the interval, and it follows that a solution $y(x)$ of (1) for which $y(a)$ and $y'(a)$ are finite for some positive a, will be continuous throughout $(0, \infty)$.

We remark that much of what follows will remain true if we set $\epsilon = 0$ in (2c), although in some cases it may then be necessary to add the condition $F(0, x) = 0$. It may also be noted that condition (2c) prevents the linear equation $y'' + p(x)y = 0$ $(p(x) > 0)$ from being included in (1). The use of this condition will thus tend to emphasize those aspects of equation (1) which are not shared by the linear equation.

A special case of equation (1) which exhibits many of the features encountered in the general case is that of the equation

$$y'' + p(x)y^{2n+1} = 0, \qquad p(x) > 0, \tag{3}$$

where $p(x)$ is continuous in $(0, \infty)$ and n is a positive integer [1; 4]. As shown in [4], the study of this equation may be based on the consideration of the homogeneous functional

$$J(y) = \left(\int_a^b y'^2 dx \right)^{n+1} \Big/ \int_a^b p y^{2n+2} dx \tag{4}$$

and its extremal values under appropriate admissibility conditions for the functions $y(x)$. In the case of the general equation (1), no corresponding single

Received by the editors April 6, 1959.

(1) This research was supported by the United States Air Force Office of Scientific Research.

functional exists and this causes certain additional complications in the variational treatment of the equation.

Our main topic will be the problem of oscillation or nonoscillation of the solutions of (1). We shall call a solution $y(x)$ of (1) *nonoscillatory* if there exists a positive number x_0 such that $y(x)\neq 0$ for $x>x_0$. If this condition does not hold, i.e., if $y(x)$ has an infinity of zeros in $(0, \infty)$, $y(x)$ will be called *oscillatory*. An equation will be termed nonoscillatory if all of its solutions are nonoscillatory. It should be noted, however, that the nonoscillation of equation (1) is not quite the same thing as the nonoscillation of a linear equation. As shown in [4] in the case of equation (3), there always exist solutions which have any desired number of zeros in any given interval, even if the equation is nonoscillatory.

2. A more penetrating study of equation (1) shows that the definition of nonoscillation just given is, for many purposes, not precise enough. A distinction must be made between solutions which *never* vanish in the entire interval of continuity of the equation, and nonoscillatory solutions which vanish there at least once. In the case of a linear equation this distinction is not necessary since, in view of the Sturm separation theorem, the existence of a solution of the first type implies the existence of an infinity of solutions of the second. This, however, is not necessarily true for a general equation (1). Here it may happen that the equation has solutions which never vanish in the interval of continuity, while every solution which vanishes once also vanishes an infinity of times. As an example for this type of behavior we consider the equation

$$y'' + x^{-2}yF(x^{-1}y^2) = 0. \tag{5}$$

Its general solution is

$$y(x) = x^{1/2}u(\log x),$$

where $u(t)$ is the general solution of

$$\ddot{u} - \frac{1}{4}u + uF(u^2) = 0. \tag{5'}$$

In accordance with conditions (2), $F(\zeta)$ increases from 0 to ∞ as ζ varies over the same interval. Hence, there exists a positive number c such that $F(c^2) = 1/4$. The function $u(t)\equiv c$ is evidently a solution of (5′) and it follows that (5) has the nonoscillatory solution $y(x)=cx^{1/2}$ (the equation has, in fact, an infinity of different nonoscillatory solutions, as can be shown either directly or as a consequence of Theorem I).

On the other hand, if $G'(\zeta)=F(\zeta)$, $G(0)=0$, and $u(t)$ is a solution of (5′) for which $u(a)=0$, we conclude from (5′) that

$$\dot{u}^2 - \frac{1}{4}u^2 + G(u^2) = \dot{u}^2(a). \tag{5''}$$

We assume that $t=a$ is the largest zero of $u(t)$ and that $\dot{u}(a)>0$. We distinguish two cases, according as $u(t)$ is ultimately larger than c, or $0<u(t)\leqq c$ in (a, ∞). In the first case we observe that the function $G(\zeta)-\zeta/4$ is convex and increasing for $\zeta>c^2$. On the other hand, (5″) shows that $G(u^2)-u^2/4$ is bounded from above, and a contradiction can be avoided only if $u(t)$ has a finite limit M as $t\to\infty$. By (5′), this implies the existence of $\lim \ddot{u}(t)$ for $t\to\infty$ and, because of the boundedness of $u(t)$, this limit must be zero. Hence, $F(M^2)=1/4$, i.e., $M=c$, which is absurd. If we assume $0<u(t)\leqq c$ throughout (a, ∞), the same argument shows that again $u(t)\to c$ for $t\to\infty$. It then follows from (5″) that $G(c^2)-c^2/4=\dot{u}^2(a)$. But this is again absurd since, for $\zeta\in(0, c)$, the function $G(\zeta)-\zeta/4$ is decreasing and this shows that $G(c^2)-c^2/4<0$. It has thus been established that every solution of (5) which vanishes once in $(0, \infty)$ vanishes an infinity of times in this interval.

3. When it will be necessary to distinguish between the two types of nonoscillatory solutions described above—those which do not change sign in the entire interval of continuity, and those which change their sign at least once—we shall call the solutions of the second type *properly nonoscillatory*. As the preceding example shows, an equation of type (1) may have nonoscillatory solutions without having any which are properly nonoscillatory. If these two classes of solutions are not kept apart, it is not difficult to derive a necessary and sufficient criterion for the existence of nonoscillatory solutions. It was shown by Atkinson [1] that equation (3) has nonoscillatory solutions if, and only if,

$$\int^{\infty} xp(x)dx < \infty \tag{6}$$

(for an extension of this result to a somewhat more general class of equations cf. [2]). The following result generalizes Atkinson's theorem to the general equation (1).

THEOREM I. *Equation* (1) *has bounded nonoscillatory solutions if, and only if,*

$$\int^{\infty} xF(c, x)dx < \infty \tag{7}$$

for some positive constant c.

If $g(x, t)=t-a$ for $t\in(a, x)$ and $g(x, t)\equiv x-a$ for $t\geqq x$, (1) may be replaced by the integral equation

$$y(x) = y(a) + \int_a^b g(x, t)y(t)F(y^2, t)dt + (x-a)y'(b), \tag{8}$$

where $0<a<x<b<\infty$. We assume that $y(x)$ is a bounded nonoscillatory solution of (1) (for trivial reasons $y(x)$ may be supposed to be ultimately positive) and that a is large enough so that $y(x)>0$ for $x \geqq a$. Since $y(x)$ is increasing for $x>a$, it follows from (8) that

$$M \geqq y(x) \geqq \alpha + \alpha \int_a^x (t-a)F(\alpha^2, t), \qquad \alpha = y(a).$$

Since x may be taken arbitrarily large, this shows that condition (7) is necessary (and also that c may be any positive number smaller than $\lim_{x\to\infty} y(x)$).

To prove sufficiency, we assume that (7) holds for $c=M^2$ and that a is taken large enough so that

$$\int_a^\infty (t-a)F(M^2, t)dt \leqq \frac{1}{2} \cdot$$

If A is a positive number such that $A \leqq M/2$, we will then have

$$A + M\int_a^\infty (t-a)F(M^2, t)dt \leqq M. \tag{9}$$

We now consider the sequence of functions $y_n(x)$ given by $y_0(x) \equiv A$,

$$y_{n+1}(x) = A + \int_a^\infty g(x, t)y_n(t)F(y_n^2, t)dt \qquad (a \leqq x < \infty), \tag{10}$$

and we observe that $A \leqq y_n(x) \leqq M$ for all n. The lower bound is evident, and the upper bound follows from (9), (10), and complete induction. Indeed, if $y_n(x) \leqq M$ we have

$$y_{n+1} \leqq A + M\int_a^\infty (t-a)F(M^2, t)dt \leqq M,$$

and $y_0(x) \equiv A < M$.

Since $|g(x_2, t) - g(x_1, t)| \leqq |x_2 - x_1|$, (10) shows that

$$| y_{n+1}(x_2) - y_{n+1}(x_1) | \leqq M | x_2 - x_1 | \int_a^\infty F(M^2, t)dt.$$

By (7), the integral exists, and we thus find that the sequence of functions $\{y_n(x)\}$ is equicontinuous in any interval $[a, b]$, where b may be taken arbitrarily large. It is moreover true that, for fixed x, the sequence $\{y_n(x)\}$ is increasing. By (10), we have

$$y_{n+1}(x) - y_n(x) = \int_a^\infty g(x, t)[y_nF(y_n^2, t) - y_{n-1}F(y_{n-1}^2, t)]dt.$$

If $y_n(x) \geqq y_{n-1}(x)$ throughout (a, ∞), it therefore follows from (2c) that $y_{n+1}(x) > y_n(x)$ in the same interval. Since $y_1(x) > y_0(x) = A$, $y_{n+1}(x) > y_n(x)$ for all n. It may, incidentally, be remarked that the choice $y_0(x)\equiv M$ for the first function would have resulted in a monotonically decreasing sequence.

It follows from all this that $y(x)=\lim_{n\to\infty} y_n(x)$ exists and is continuous in any interval $[a, b]$. If we write (10) in the form

$$y_{n+1}(x) = \int_a^b g(x, t)y_n(t)F(y_n^2, t)dt + R(b)$$

and observe that, in view of (7), the bound

$$M\int_b^\infty (t - a)F(M^2, t)dt$$

for $R(b)$ can be made arbitrarily small by taking b sufficiently large, we arrive at

$$\left| y_{n+1}(x) - A - \int_a^b g(x, t)y_nF(y_n^2, t)dt\right| \leqq \epsilon$$

where $\epsilon\to 0$ for $b\to\infty$ and ϵ is independent of n. Passing to the limit, we obtain

$$\left| y(x) - A - \int_a^b g(x, t)yF(y^2, t)dt\right| \leqq \epsilon,$$

which shows that $y(x)$ satisfies the identity

$$y(x) = A + \int_a^\infty g(x, t)yF(y^2, t)dt$$

for all x in $[a, \infty)$. Since

$$y(x) - y(x_0) = \int_{x_0}^x (t - x_0)yF(y^2, t)dt + (x - x_0)\int_x^\infty yF(y^2, t)dt,$$

$y(x)$ is differentiable, and $y'(x)=\int_x^\infty yF(y^2, t)dt$. In view of

$$y'(x) - y'(x_0) = -\int_{x_0}^x y(t)F(y^2, t)dt,$$

$y(x)$ also has a second derivative, and we have proved that $y(x)$ is a solution of (1). This establishes the sufficiency of condition (7). It may be noted that our argument proves the existence of an infinity of different nonoscillatory solutions of (1) since A could be any number such that $0<A<M/2$ (a slight modification of the argument shows that A may be any number such that $0<A<M$).

4. As the example of equation (5) shows, the condition (7) is not sufficient to guarantee the existence of a properly nonoscillatory solution. Before taking up the discussion of either necessary or sufficient conditions for the existence

of such solutions which are "sharp" in some sense, we consider a condition which not only ensures the existence of solutions of this type but also gives a complete description of their asymptotic behavior.

THEOREM II. *If*

$$\int^{\infty} tF(ct^2, t)dt < \infty \tag{11}$$

for some positive c, equation (1) *has properly nonoscillatory solutions. If* (11) *holds for all positive c, all nonoscillatory solutions of* (1) *are either bounded or $\sim\beta x$ ($\beta = const.$) for large x.*

Before proving this result we remark that condition (11) is necessary for the existence of a solution of the second type, since it follows from $y(x) \geqq \alpha x$ $(\alpha > 0)$ that

$$y'(x_0) = y'(x) + \int_{x_0}^{x} yF(y^2, t)dt \geqq \alpha \int_{x_0}^{x} tF(\alpha^2 t^2, t)dt$$

for all x larger than x_0. If $F(\eta, x)$ is such that (11) holds for all positive constants c provided it is true for some particular c, (11) may therefore be replaced by the assumption that there exists a solution which is $\sim\beta x$. An example of such a function $F(\eta, x)$ is

$$F(\eta, x) = \sum_{\nu=1}^{m} p_\nu(x)\eta^\nu, \qquad p_\nu(x) \geqq 0,$$

which, in view of $F(c\eta, x) \leqq c^m F(\eta, x)$ $(c > 1)$, has the required property. For such functions $F(\eta, x)$, Theorem II has the following corollary.

COROLLARY. *If, for $c > 1$,*

$$F(c\eta, t) \leqq AF(\eta, t),$$

where A is independent of η and t, and if equation (1) *has a solution which is $\sim\beta x$ for large x, then all nonoscillatory solutions of* (1) *are either bounded or such that $x^{-1}y(x)$ has a finite nonzero limit as $x \to \infty$.*

We now turn to the proof of Theorem II. Because of (11) we can choose a value a such that

$$\int_a^{\infty} tF(ct^2, t)dt < 1. \tag{12}$$

We denote by $y(x)$ the solution of (1) determined by $y(a) = 0$, $0 < y'(a) = \alpha \leqq c^{1/2}$ and we assume that $y(x)$ vanishes at some point in (a, ∞). In this case there exists a point $x = b$ in (a, ∞) such that $y'(b) = 0$ and $y(x) > 0$ in $(a, b]$. By (1), we have

$$y(b) = \int_a^b (t - a)y(t)F(y^2, t)dt.$$

Since $0 < y(t) \leqq y(b)$ and $y(t) \leqq y'(a)(t-a) \leqq c^{1/2}(t-a)$ in $(a, b]$, it follows that

$$1 \leqq \int_a^b (t - a)F[c(t - a)^2, t]dt,$$

and this contradicts (12). This shows that $y'(x)$ must remain positive throughout (a, ∞), and thus proves the first half of Theorem II.

To prove the other half, we replace (1) by the equivalent integral equation

$$y(x) = y(a) + \int_a^x (t - a)y(t)F(y^2, t)dt + (x - a)y'(x). \tag{13}$$

If $y(x)$ is a nonoscillatory solution, we choose a large enough so that $y(x) > 0$ for $x > a$. Since $y''(x) < 0$ for $x \geqq a$, there exists a constant α such that $y(x) \leqq \alpha x$ in (a, ∞). By (11), a may be taken large enough so that

$$\int_a^\infty tF(\alpha^2 t^2, t)dt < \frac{\epsilon}{2} \tag{14}$$

where ϵ is arbitrarily small. Since

$$\int_a^x (t - a)y(t)F(y^2, t)dt \leqq y(x) \int_a^x tF(\alpha^2 t^2, t)dt \leqq \frac{1}{2}\epsilon y(x),$$

it follows from (13) that

$$1 \leqq \frac{y(a)}{y(x)} + \frac{1}{2}\epsilon + \frac{xy'(x)}{y(x)} \cdot$$

If $y(x)$ is not bounded in (a, ∞), the first term on the right-hand side can be made smaller than $\epsilon/2$ by choosing sufficiently large values of x. For such values we thus have $\liminf xy'y^{-1} \geqq 1 - \epsilon$ for $x \to \infty$. Since, on the other hand, $(xy' - y)' = xy'' < 0$, we have $xy' - y <$ const., i.e. $\limsup xy'y^{-1} \leqq 1$. Hence, $\lim xy'y^{-1}$ exists and is equal to 1.

To show that this is equivalent to $y(x) \sim \beta x$, we take b large enough so that $xy' \geqq y(1 - \epsilon)$ in (b, ∞). We then have

$$\begin{aligned}(1 - \epsilon)[y'(b) - y'(x)] = (1 - \epsilon) \int_b^x yF(y^2, t)dt &\leqq y'(b) \int_b^x tF(\alpha^2 t^2, t)dt \\ &\leqq \frac{\epsilon}{2} y'(b),\end{aligned}$$

the last inequality following from (14). Hence,

$$(1-\epsilon)y'(x) \geqq \left(1 - \frac{3}{2}\epsilon\right)y'(b)$$

for all x larger than b. This shows that $y'(x)$ has a positive lower bound as $x \to \infty$. But $y'(x)$ is monotonically decreasing as x increases, and it follows that $y'(x)$ tends to a positive limit as $x \to \infty$. Since $xy'(x)[y(x)]^{-1} \to 1$ for $x \to \infty$, this completes the proof of Theorem II.

5. Theorem II does not assert that, under the given hypotheses, *all* solutions of (1) are nonoscillatory and, indeed, it is doubtful whether this is true if $F(\eta, x)$ is only subject to the conditions (2). As the next result shows, oscillatory solutions can be excluded by an additional monotonicity assumption.

THEOREM III. *If* (11) *holds for all positive values of c and if, in addition to* (2), $F(\eta, x)$ *satisfies the condition*

$$F(\eta, x_2) \leqq F(\eta, x_1) \qquad \text{for } 0 < x_1 < x_2 < \infty \tag{15}$$

and all positive η, *all solutions of* (1) *are nonoscillatory.*

In the case of equation (3), this result was proved by Atkinson [1]. To obtain the general result, we need the following lemma.

LEMMA. *Let* $y(x)$ *be a solution of* (1), *where* $F(\eta, x)$ *is subject to conditions* (2) *and* (15). *If* $G(\eta, x) = \int_0^\eta F(t, x)dt$, *the expression*

$$y'^2(x) + G[y^2(x), x] \tag{16}$$

is a nonincreasing function of x.

Since $F(\eta, t)$ increases with η (for fixed x), $G(\eta, x)$ is a convex function of η and we have $G(\zeta, x) \geqq G(\eta, x) + (\zeta - \eta)F(\eta, x)$ for arbitrary positive values of ζ and η. Using this, and the fact that $G(\eta, x)$ also has the monotonicity property (15), we obtain

$$\begin{aligned} &G[y^2(a), a] - G[y^2(b), b] \\ &\qquad \geqq G[y^2(b), a] + [y^2(a) - y^2(b)]F[y^2(b), a] - G[y^2(b), b] \\ &\qquad \geqq [y^2(a) - y^2(b)]F[y^2(b), a], \end{aligned}$$

provided $0 < a < b < \infty$. If $a = x_0 < x_1 < \cdots < x_n = b$, it thus follows that

$$\begin{aligned} G[y^2(b), b] - G[y^2(a), a] &\leqq \sum_{\nu=1}^{n} [y^2(x_{\nu+1}) - y^2(x_\nu)]F[y^2(x_{\nu+1}), x_\nu] \\ &= 2\sum_{\nu=1}^{n} y(x_\nu^*)y'(x_\nu^*)F[y^2(x_{\nu+1}), x_\nu](x_{\nu+1} - x_\nu), \end{aligned}$$

where $x_\nu \leqq x_\nu^* \leqq x_{\nu+1}$. Passing to the limit, we obtain

$$G[y^2(b), b] - G(y^2(a), a] \leqq 2\int_a^b yy'F(y^2, x)dx$$
$$= -2\int_a^b y'y''dx = y'^2(a) - y'^2(b),$$

and this proves the lemma.

To prove Theorem III, we assume that $y(x)$ is an oscillatory solution of (1) and that $x = a_\nu$ $(0 < a_1 < a_2 < \cdots)$ is a sequence of its zeros. In view of $G(0, x) = 0$, the lemma shows that $y'^2(a_{\nu+1}) \leqq y'^2(a_\nu) \leqq y'^2(a_1) = \alpha^2$. The integral (11) converges for all positive c and we may therefore find a finite number β such that

$$\int_\beta^\infty tF(\alpha^2 t, t)dt < 1.$$

But, as shown in the proof of Theorem II, this implies that a solution $y(x)$ such that $y(\beta) = 0$, $[y'(\beta)]^2 \leqq \alpha^2$ cannot vanish in (β, ∞). The number β may be identified with a sufficiently large zero of $y(x)$, and our argument is thus seen to lead to a conclusion which is incompatible with the assumption that $y(x)$ is oscillatory. This proves Theorem III.

An example of a class of equations satisfying the hypotheses of Theorem III is

$$y'' + x^{-2-\nu}yF(x^{-2}y^2) = 0 \qquad (\nu > 0),$$

where $F(0) = 0$ and F is a nondecreasing function of its argument. A particularly instructive case is obtained for $\nu = 2$. Here the general solution is of the form

$$y(x) = xu(a^{-1} - x^{-1}), \qquad a > 0, \tag{17}$$

where $u(t)$ is any solution of

$$\ddot{u} + uF(u^2) = 0$$

for which $u(0) = 0$. It is easy to see that all solutions of the latter equation are periodic and that the distance between consecutive zeros decreases as $\dot{u}(0)$ increases. If a^{-1} does not coincide with a zero of $u(t)$, (17) shows that $y(x) \sim u(a^{-1})x$ for large x. If $u(a^{-1}) = 0$, it follows from (17) that $y(x) \to \dot{u}(a^{-1})$ as $x \to \infty$. Both types of solutions permitted by Theorem II are thus represented. For any given positive a, there exists a discrete infinity of bounded solutions which vanish at a; these solutions are uniquely determined by the number of zeros they have in (a, ∞).

6. The results obtained so far show that a rather complete description of the oscillatory behavior of an equation of type (1) is available (a) if condition (11) holds, and (b) if condition (7) does *not* hold. The oscillation problem,

properly so called, of equation (1) refers to the question as to the oscillatory behavior of the equation if (7) holds, but (11) does not. It may be pointed out that this problem is not a mere extension of the oscillation problem for the linear equation $y''+p(x)y=0$ ($p(x)>0$). In the case of the latter equation, conditions (7) and (11) coincide and there thus exists no analogue to the type of difficulty met in the case of the general equation (1).

Another feature which is absent from the linear case is the fact that, for any two numbers a, b such that $0<a<b<\infty$, (1) always has solutions which vanish at $x=a$, $x=b$ and are $\neq 0$ in (a, b). The existence of such solutions will be obtained as a by-product of the variational treatment of (1) to be given presently, but it may also be established by an elementary continuity argument. We remark that we shall concentrate on the boundary conditions $y(a)=y'(b)=0$ rather than $y(a)=y(b)=0$, since these solutions are of greater relevance for the oscillation problem. The corresponding results for the condition $y(a)=y(b)=0$ may be obtained by a trivial modification of the argument.

The variational problem most suitable for a more penetrating study of the oscillation properties of (1) is

$$J(y) \equiv \int_a^b [y'^2 - G(y^2, x)]dx = \min = \lambda(a, b), \tag{18}$$

where $y(x)$ is subject to the admissibility conditions

$$y(a) = 0, \qquad y(x) \not\equiv 0, \qquad y(x) \in D^1 \text{ in } [a, b], \tag{19}$$

and the normalization condition

$$\int_a^b y'^2 dx = \int_a^b y^2 F(y^2, x)dx. \tag{20}$$

The function G appearing in (18) is defined, as before, by

$$G(\eta, x) = \int_0^\eta F(t, x)dt. \tag{21}$$

We also remark that, in view of (2b) and (2c), any function $y(x)$ satisfying (19) can be multiplied by a positive constant α such that $\alpha y(x)$ satisfies both (19) and (20).

We first show that our variational problem has a solution. By (2c) and (21), we have

$$G(\eta, x) = \int_0^\eta t^\epsilon [t^{-\epsilon} F(t, x)]dt \leqq \eta^{-\epsilon} F(\eta, x) \int_0^\eta t^\epsilon dt,$$

i.e.,

$$G(\eta, x) \leqq (1 + \epsilon)^{-1} \eta F(\eta, x).$$

Hence,

$$\eta F(\eta, x) - G(\eta, x) \geqq \epsilon(1+\epsilon)^{-1}\eta F(\eta, x)$$

and thus, by (18) and (20),

$$J(y) \geqq \epsilon(1+\epsilon)^{-1}\int_a^b y'^2 dx. \tag{22}$$

It may be noted in passing that the proof of (22) is the only occasion at which the existence of a positive ϵ in condition (2c) is required. All results which do not depend on (22) remain true if (2c) is only assumed to hold for $\epsilon=0$.

It follows from (22) that for a sequence of functions $y(x)$ for which $J(y)$ tends to its greatest lower bound $\lambda(a, b)$, we have

$$\int_a^b y'^2 dx < M < \infty,$$

where M is a suitable constant. Since, moreover, $y(a)=0$, the sequence is equicontinuous and we may conclude that, for a subsequence $y_n(x)$, we have $J(y_n)\to\lambda(a, b)$ and $y_n(x)\to y_0(x)$ uniformly in $[a, b]$. In order to prove that this continuous function $y_0(x)$ indeed solves our problem it remains to be shown that $y_0(x)\in D^1$ in $[a, b]$ and that $\int_a^b y_n'^2$ converges to $\int_a^b y_0'^2 dx$.

If $y(x)$ is any of the functions of the minimal sequence $\{y_n(x)\}$, we define an associated function $u(x)$ by

$$u''(x) = -\alpha y(x)F(y^2, x), \qquad u(a) = u'(b) = 0, \tag{23}$$

where the positive number α is determined by the normalization condition (20), i.e.,

$$\int_a^b u'^2 dx = \int_a^b u^2 F(u^2, x)dx. \tag{24}$$

As pointed out above, such a number α always exists. By (20), (23), and (24), we have

$$\begin{aligned}\alpha^2\left(\int_a^b y^2F(y^2, x)dx\right)^2 &= \left(\int_a^b yu''dx\right)^2 = \left(\int_a^b y'u'dx\right)^2\\ &\leqq \int_a^b u'^2dx\int_a^b y'^2dx = \int_a^b u^2F(u^2, x)dx\int_a^b y^2F(y^2, x)dx,\end{aligned}$$

i.e.,

$$\alpha^2\int_a^b y^2F(y^2, x)dx \leqq \int_a^b u^2F(u^2, x)dx. \tag{25}$$

On the other hand, again by (23) and (24),

$$\left(\int_a^b u^2F(u^2, x)dx\right)^2 = \left(\int_a^b u'^2dx\right)^2 = \left(\int_a^b uu''dx\right)^2$$
$$= \alpha^2\left(\int_a^b uyF(y^2, x)dx\right)^2 \leqq \alpha^2\int_a^b u^2F(y^2, x)dx\int_a^b y^2F(y^2, x)dx.$$

In view of (25), this leads to

$$\int_a^b u^2F(u^2, x)dx \leqq \int_a^b u^2F(y^2, x)dx. \tag{26}$$

By (2c), $F(\eta, x)$ is an increasing function of η, and it follows that the function $G(\eta, x)$ defined in (21) is convex in η. Hence,

$$\int_a^b G(u^2, x)dx \geqq \int_a^b G(y^2, x)dx + \int_a^b (u^2 - y^2)F(y^2, x)dx$$

and, if this is combined with (26),

$$\int_a^b [u^2F(u^2, x) - G(u^2, x)]dx \leqq \int_a^b [y^2F(y^2, x) - G(y^2, x)]dx.$$

If we utilize (20), (24), and (18), we finally obtain

$$J(u) \leqq J(y). \tag{27}$$

This shows that the original minimal sequence $\{y_n(x)\}$ may be replaced by a minimal sequence $\{u_n(x)\}$, where each $u(x)$ is obtained from the corresponding $y(x)$ by means of (23) and (24). Since $y_n(x)$ was shown to tend to a continuous limit function, this implies that the same is true of the sequence $\{u_n''(x)\}$. Hence $u_n'(x) = -\int_x^b u_n''dx$ and $u_n(x) = \int_a^x u_n'dx$ likewise tend to continuous limits, and it follows that $u_n(x) \to u_0(x)$, where $u_0(x) \in C^2[a, b]$, and $J(u_n) \to J(u_0) = \lambda(a, b)$. The function $u_0(x)$ is thus shown to be the solution of our minimum problem.

We note that $u_0(x)$ cannot be identically zero. Indeed, if $\beta = \int_a^b u'^2dx$ and $u(x) \not\equiv 0$, we have

$$u^2(x) = \left(\int_a^b u'dx\right)^2 \leqq \beta(x - a),$$

whence, in view of (24) and the fact that $\beta > 0$,

$$1 \leqq \int_a^b (x - a)F[\beta(x - a), x]dx,$$

and this makes it evident that β has a positive lower limit β_0. Hence, $\int_a^b u_0'^2dx \geqq \beta_0$ and, therefore, $u_0(x) \not\equiv 0$. Furthermore, $u_0(x)$ must be—if normalized by the condition $u'(a) \geqq 0$—a positive, increasing and concave function of x in $(a, b]$. To see this, we note that (18), (19), and (20) remain unchanged if $y(x)$

is replaced by $|y(x)|$, and that it is therefore sufficient to consider functions $y(x)$ which are non-negative in $[a, b]$. In view of (23), we then have $u''(x) \leq 0$, $u'(x) = \alpha\int_x^b yF(y^2, x)dx > 0$, $u(x) = \alpha\int_a^x u'(x)dx > 0$, and the assertion follows.

In view of the circumstances under which the Schwartz inequality was used in the proof of (27), the sign of equality in (27) is possible only if $y(x)$ and $u(x)$ coincide. (23) shows that, in this case, $y(x)$ must be a solution of

$$u'' + \alpha uF(u^2, x) = 0.$$

Because of $u(a) = u'(b) = 0$, it follows that

$$\int_a^b u'^2 dx = \alpha \int_a^b u^2 F(u^2, x)dx,$$

and a comparison with (24) shows that we must have $\alpha = 1$. Equality in (27) is therefore possible only if $y(x)$ is a solution of (1) for which $y(a) = y'(b) = 0$.

If the transformation (23) is applied to $y(x) = u_0(x)$, it follows from the minimum property of $u_0(x)$ that in this case we necessarily have equality in (27). In view of the foregoing, this proves that $u_0(x)$ is a solution of (1). For convenient reference, we state the results of this section as a theorem.

THEOREM IV. *The variational problem* (18) *with the side conditions* (19) *and* (20) *is solved by a solution* $y(x)$ *of* (1) *for which* $y(a) = y'(b) = 0$ *and* $y(x) > 0$ *in* $(a, b]$.

The treatment of the oscillation problem of equation (1) would be very much easier if it could be shown that these properties determine the solution $y(x)$ of (1) uniquely. Whether or not we have uniqueness of this kind remains an open question, although certain doubts may be aroused by an example exhibited in [4] of three distinct solutions of $y'' + p(x)y^3 = 0$ all of which vanish at $x = a$ and $x = b$, and are positive in (a, b). There are, however, certain indications that such an event is less likely to happen in the case of the boundary conditions $y(a) = y'(b) = 0$, and that certain comparatively mild restrictions—such as, possibly, (15)—would prevent it in either case.

It may be noted that the assumption $y(x) \not\equiv 0$ in (19) is necessary in order to obtain the solution of the variational problem described in Theorem IV. If this assumption is dropped, the problem has the trivial solution $y(x) \equiv 0$.

7. The usefulness of Theorem IV for the treatment of the oscillation problem of (1) is due to the following monotonicity property of the minimum value of $J(y)$.

THEOREM V. *If* $0 < a' < a < b < b' < \infty$ *and* $\lambda(a, b)$ *denotes the function defined in* (18), *then*

$$\lambda(a', b') < \lambda(a, b), \tag{28}$$

unless $a' = a$ *and* $b' = b$.

If $b=b'$, (28) is obvious since the function $y(x)$ which solves the variational problem (18) for the interval $[a, b]$ is also an admissible function in the corresponding problem for the interval $[a', b]$ if we define $y(x)\equiv 0$ in $[a', a]$. It is thus sufficient to prove (28) under the assumption $a'=a$.

If $y(x)$ is the function described in Theorem IV, we define a function $y_1(x)$ as follows: $y_1(x)=y(x)$ in $[a, b]$; $y_1(x)\equiv y(b)$ in $[b, b']$. As pointed out above in a similar situation, there exists a positive constant γ such that

$$\int_a^{b'} y_1'^2 dx = \int_a^{b'} y_1^2 F(\gamma y_1^2, x)dx.$$

The function $y^*(x)=\gamma^{1/2}y_1(x)$ will then be normalized as in (20), and it follows from Theorem IV that

$$\lambda(a, b') < J_{b'}(y^*). \tag{29}$$

On the other hand,

$$\begin{aligned} J_{b'}(y^*) &= \int_a^{b'} [y^{*\prime 2} - G(y^{*2}, x)]dx \\ &= \gamma \int_a^b y'^2 dx - \int_a^{b'} G(y^{*2}, x)dx \\ &< \gamma \int_a^b y'^2 dx - \int_a^b G(y^{*2}, x)dx \\ &= \gamma \int_a^b y'^2 dx - \int_a^b G(\gamma y^2, x)dx. \end{aligned}$$

Since $G(t, x)$ is a convex function of t and $G_t(t, x)=F(t, x)$, we have

$$\int_a^b G(\gamma y^2, x)dx \geqq \int_a^b G(y^2, x)dx + (\gamma - 1)\int_a^b y^2F(y^2, x)dx, \tag{30}$$

whence

$$J_{b'}(y^*) < \gamma \int_a^b y'^2 dx - \int_a^b G(y^2, x)dx + (1-\gamma)\int_a^b y^2F(y^2, x)dx.$$

Because of (20), this is equivalent to

$$J_{b'}(y^*) < \int_a^b [y'^2 - G(y^2, x)]dx = J_b(y) = \lambda(a, b).$$

In view of (29), this completes the proof of Theorem V.

An application of inequality (22) to the function $y(x)$ of Theorem IV shows that Theorem V has the following corollary.

COROLLARY. *If* $0<a<a_0<b_0<b<\infty$ *and* $y(x)$ *denotes the solution of* (1) *whose existence is established by Theorem* IV, *then*

$$\int_a^b y'^2 dx \leqq M = M(a_0, b_0), \tag{31}$$

where M *is independent of* a *and* b.

An argument similar to that used in the proof of Theorem V establishes the following comparison theorem.

THEOREM VI. *If*

$$F(t, x) \leqq F_1(t, x) \tag{32}$$

for all positive t *and* x, *and if* $\lambda(a, b)$ *and* $\lambda_1(a, b)$ *are defined by* (18) *for the equations*

$$u'' + uF(u^2, x) = 0$$

and

$$v'' + vF_1(v^2, x) = 0 \tag{33}$$

respectively, then

$$\lambda_1(a, b) \leqq \lambda(a, b).$$

If γ is defined by

$$\int_a^b u'^2 dx = \int_a^b u^2 F_1(\gamma u^2, x)dx,$$

the function $w(x) = \gamma^{1/2}u(x)$ has the normalization (20) (with respect to equation (33)), and we must have

$$\lambda_1(a, b) \leqq \int_a^b [w'^2 - G_1(w^2, x)]dx.$$

(21) and (32) show that $G(w^2, x) \leqq G_1(w^2, x)$. In view of (30), we thus obtain

$$\begin{aligned}
\lambda_1(a, b) &\leqq \gamma \int_a^b u'^2 dx - \int_a^b G(\gamma u^2, x)dx \\
&\leqq \gamma \int_a^b u'^2 dx - \int_a^b G(u^2, x)dx - (\gamma - 1)\int_a^b u^2 F(u^2, x)dx \\
&= \int_a^b [u'^2 - G(u^2, x)]dx \\
&= \lambda(a, b).
\end{aligned}$$

8. According to Theorem V, $\lim_{b\to\infty} \lambda(a, b)=\lambda(a)$ exists, and is either positive or zero. Theorem V also shows that $\lambda(a)$ is a nondecreasing function of a. We shall call $\lambda(a)$ the *characteristic number* of equation (1) for the point $x=a$. The following result points up the connection between $\lambda(a)$ and the oscillatory behavior of equation (1).

THEOREM VII. *If the solution $y(x)$ of* (1) *determined by $y(a)=y'(b)=0$, $y(x)>0$ $(x\in(a, b])$ is unique, and if equation* (1) *has a properly nonoscillatory solution $y_0(x)$ whose last zero is at $x=a$, then the characteristic number $\lambda(a)$ must be positive.*

If $y(x, \alpha)$ is the solution of (1) defined by $y(a)=0$, $y'(a)=\alpha>0$, it follows from the existence theorem that the location of the first zero of $y'(x, \alpha)$ in (a, ∞) varies continuously with α. By Theorem IV this zero, say $x=b$, will take all values in (a, ∞) if α goes through the positive numbers. By assumption the correspondence between α and b is one-to-one. Since $y_0'(x)>0$ in (a, ∞) it follows that the values of α to which there correspond values of b are either all larger or all smaller than $y_0'(a)$. To exclude the second alternative, we show that α becomes arbitrarily large if b approaches a. By (1), we have

$$y'(a) = \int_a^b yF(y^2, x)dx.$$

Since $y(x)\leqq y'(a)(x-a)=\alpha(x-a)$ in (a, b), it follows that

$$1 \leqq \int_a^b (x - a)F(\alpha^2(x - a)^2, x)dx,$$

and this makes it evident that α cannot remain bounded if b tends to a. We may thus conclude that $\alpha\geqq y_0'(a)$.

Suppose now that $\lambda(a)=0$. In view of the definition of $\lambda(a)$, this implies that the functional (18) can be made arbitrarily small by taking b large enough. Because of (22) we can, moreover, choose b sufficiently large so that $\int_a^b y'^2dx\leqq\delta^2$ where δ is a given small number and $y'(b)=0$. For this function $y(x)$ we have

$$y^2(x) \leqq (x - a)\int_a^b y'^2dx \leqq \delta^2(x - a)$$

for $x\in(a, b)$. If c is a fixed number in (a, b), it then follows from

$$y'(a) = y'(c) + \int_a^c yF(y^2, x)dx$$

and $y'(c)(c-a)<y(c)<\delta(c-a)^{1/2}$ that

$$\alpha = y'(a) < \delta(c - a)^{-1/2} + \delta\int_a^c (x - a)^{1/2}F(\delta^2(x - a), x)dx.$$

This shows that, under the assumption $\lambda(a)=0$, we have $\alpha\to 0$ for $b\to\infty$. But this contradicts the inequality $\alpha\geqq y_0'(a)$, and the proof of Theorem VII follows.

It will become apparent in the sequel that, although the condition $\lambda(a)>0$ is not equivalent to the existence of a properly nonoscillatory solution of (1) whose last zero is located at $x=a$, the two conditions—i.e. $\lambda(a)>0$ and the existence of such a solution—are not very far removed from each other. Theorem VII shows that, under the uniqueness assumption made, $\lambda(a)>0$ is the weaker condition. This is also illustrated by equation (5) which was shown to have no properly nonoscillatory solutions, but for which, according to Theorem VIII, $\lambda(a)>0$ for all positive a. On the other hand, condition (40) which will be shown to be sufficient to produce properly nonoscillatory solutions of (1) is not very much stronger than condition (37) which is equivalent to $\lambda(a)>0$.

As a first step towards the proof of Theorem VIII we derive the following lemma.

Lemma. *If* $\lambda(a)=\lambda>0$, $y(a)=0$, $y(x)\in D^1$ *in* $[a,\infty)$, *and*

$$\beta=\int_a^\infty y'^2dx$$

exists, then

$$\int_a^\infty y^2F(\beta^{-1}\lambda y^2,x)dx\leqq\beta. \tag{34}$$

Proof. Let $b\in(a,\infty)$ and let α be determined by

$$\int_a^b y'^2dx=\int_a^b y^2F(\alpha y^2,x)dx. \tag{35}$$

The function $u(x)=\alpha^{1/2}y(x)$ will then have the normalization (20), and it follows from Theorems IV and V that

$$\lambda=\lambda(a)\leqq\lambda(a,b)\leqq\int_a^b[u'^2-G(u^2,x)]dx\leqq\int_a^b u'^2dx=\alpha\int_a^b y'^2dx\leqq\alpha\beta.$$

Using this inequality to estimate α in (35), we arrive at (34).

Any function $y(x)$ which satisfies the hypotheses of the Lemma will thus give rise to a condition which must hold if $\lambda(a)>0$. As an example for the type of criterion obtainable in this way we set $y(x)=(x-a)(c-a)^{-1}$ in $[a,c]$ and $y(x)=(x-a)^{\nu/2}(c-a)^{-\nu/2}$ in $[c,\infty)$, where $0\leqq\nu<1$ and $a<c<\infty$. An elementary computation shows that this choice of $y(x)$ leads to the following result.

If $\lambda=\lambda(a)>0$, $a<c<\infty$, *and* $0\leqq\nu<1$, *then*

$$\int_c^\infty (x-a)^\nu F[\lambda\gamma(x-a)^\nu, x]dx < \gamma^{-1}, \tag{36}$$

where $\gamma = 4(1-\nu)(c-a)^{1-\nu}(2-\nu)^{-2}$.

It may be noted that the integral on the left-hand side of (36) will not necessarily exist, for any positive γ, if $\nu=1$. This is shown by equation (5) for which, as already mentioned, $\lambda(a)>0$. By a proper utilization of (34) it is, however, possible to derive a stronger necessary condition which, as the following theorem shows, is also sufficient.

THEOREM VIII. *In order that, for some positive* a, $\lambda(a)>0$, *it is necessary and sufficient that there exist two positive constants* m *and* M *so that*

$$x\int_x^\infty F(mt, t)dt \leqq M \tag{37}$$

for sufficiently large x.

In the case of the necessary condition, suitable values for m and M are given in the following statement.

If $\lambda=\lambda(a)>0$, $a<x$, *and* $0<\mu<1$, *then*

$$(x-a)\int_x^\infty F(\lambda\mu(t-a), t]dt \leqq [\mu(1-\mu)]^{-1}. \tag{38}$$

We first prove (38). Applying the Lemma to the function $y(x)$ defined by $y(x)=(x-a)(b-a)^{-1/2}$ in $[a, b]$ and by $y(x)=(b-a)^{1/2}$ in $[b, \infty)$, we have, by (34),

$$b-a \geqq \int_a^b (x-a)^2F[\lambda(b-a)^{-1}(x-a)^2, x]dx.$$

If $0<\mu<1$ and $c=a+\mu(b-a)$, it follows that

$$\begin{aligned}(b-a) &\geqq \int_c^b (x-a)^2F[\lambda(b-a)^{-1}(x-a)^2, x]dx \\ &\geqq (c-a)^2\int_c^b F[\lambda\mu(x-a), x]dx\end{aligned}$$

and therefore

$$1 \geqq \mu(c-a)\int_c^b F[\lambda\mu(x-a), x]dx.$$

We now define a sequence of numbers b_ν ($\nu=0, 1, 2, \cdots$) by $\mu^\nu(b_\nu-a)=c-a$ and apply the last inequality to the interval $(b_\nu, b_{\nu+1})$. This yields

$$1 \geqq \mu^{-\nu+1}(c-a)\int_{b_\nu}^{b_{\nu+1}} F[\mu\lambda(x-a), x]dx, \qquad \nu = 0, 1, \cdots.$$

In view of

$$\int_c^\infty F[\mu\lambda(x-a), x]dx = \sum_{\nu=0}^\infty \int_{b_\nu}^{b_{\nu+1}} F[\mu\lambda(x-a), x]dx,$$

this implies (38).

We now show that condition (37) is sufficient. If $y(x)$ is the solution of (1) described in Theorem IV, we have

$$\alpha = \int_a^b y'^2 dx = \int_a^b y^2 F(y^2, x)dx \leqq \int_a^b y^2 F[\alpha(x-a), x]dx, \tag{39}$$

the last step following from

$$y^2(x) = \left(\int_a^x y'dx\right)^2 \leqq (x-a)\int_a^x y'^2 dx.$$

If $\lambda(a)=0$ we can, according to (18) and (22), make α arbitrarily small by taking b large enough. We take b sufficiently large so that $\alpha \leqq m$ where m is the constant appearing in (37), and we observe that, by (2c), we then have

$$\begin{aligned}\phi(x) \equiv \int_x^\infty F[\alpha(x-a), x]dx &\leqq (m^{-1}\alpha)^\epsilon \int_x^\infty F[m(x-a), x]dx \\ &\leqq (m^{-1}\alpha)^\epsilon x^{-1}M.\end{aligned}$$

Combining this with (39), we obtain

$$\begin{aligned}\alpha &\leqq -\int_a^b y^2\phi'(x)dx = -y^2(b)\phi(b) + 2\int_a^b yy'\phi(x)dx \\ &\leqq 2M(m^{-1}\alpha)^\epsilon \int_a^b yy'x^{-1}dx,\end{aligned}$$

and therefore

$$\begin{aligned}\alpha^2 &\leqq 4M^2(m^{-1}\alpha)^{2\epsilon}\int_a^b x^{-2}y^2dx \int_a^b y'^2dx \\ &= 4\alpha M^2(m^{-1}\alpha)^{2\epsilon}\int_a^b x^{-2}y^2dx.\end{aligned}$$

Since $y(a)=0$, it follows from a well-known inequality [3, p. 175] that

$$\int_a^b y^2 x^{-2} dx < 4 \int_a^b y'^2 dx = 4\alpha,$$

and we arrive at the inequality

$$1 \leqq 4M(m^{-1}\alpha)^{\epsilon}.$$

Since ϵ is a fixed positive number the assumption $\alpha \to 0$ is thus seen to lead to a contradiction. Hence $\lambda(a) > 0$, and Theorem VIII is proved.

It may be noted that in the case of an equation of the form (3) the constants appearing in these estimates can be determined with greater precision. If we set

$$A = \limsup_{n \to \infty} x \int_x^{\infty} p(x) x^n dx$$

and observe that, in this case $\lambda = n(n-1)^{-1}\alpha$, we obtain the estimates

$$e^{-1} < n^n(n+1)^{-n} \leqq 4A\lambda^n \leqq (n+1)^{n+1}(n-1)^{-n+1} < e^2(n+1)^2$$

for $n > 1$. If $n = 1$, i.e., in the case of the equation $y'' + py^3 = 0$, the result is

$$\frac{1}{2} \leqq 4A\lambda \leqq 1.$$

9. As already pointed out, the condition (37) is not sufficient to produce properly nonoscillatory solutions of (1). As the following theorem shows, the slightly stronger condition (40) will guarantee the existence of solutions which are properly nonoscillatory and, moreover, bounded.

THEOREM IX. *If*

$$\int^{\infty} xF(\beta x, x) dx < \infty \tag{40}$$

for all positive β, equation (1) *has bounded, properly nonoscillatory solutions. For every positive a, there exists a solution of this type whose last zero occurs at $x = a$.*

Proof. We first note that, since (40) implies (37), we have $\lambda(a, b) > \lambda(a) > 0$ for all $0 < a < b < \infty$. We denote by $y(x)$ the solution of (1) described in Theorem IV, and we observe that, by the corollary to Theorem V, there exists a constant $\beta = \beta(a, c_0)$ such that

$$\int_a^b y'^2 dx \leqq \beta \tag{41}$$

if a and c_0 are kept fixed and $a < c_0 < c < b$. Since $y'(b) = 0$, it follows from (1) that

$$y(b) = y(c) + \int_c^b (x - c)yF(y^2, x)dx.$$

Since $y(x)$ is increasing in (a, b), this implies

$$y(b) \leqq y(c) + y(b) \int_c^b xF(y^2, x)dx.$$

By (41), we have $y^2(x) \leqq \beta(x-a) < \beta x$, and thus

$$y(b) \leqq \beta^{1/2}(c - a)^{1/2} + y(b) \int_c^\infty xF(\beta x, x)dx.$$

In view of (40), the integral on the right-hand side can be made smaller than 1/2 by taking c large enough. For such values of c we then have $y(b) \leqq 2\beta^{1/2}(c-a)^{1/2}$. If we now keep c fixed and let b grow, we find that there exists a constant m such that, for $x \in (a, b)$, $y(x) \leqq y(b) \leqq m$. If $c < b_0 < b$, all these functions are thus uniformly bounded in (a, b_0) and their limit (or limits) for $b \rightarrow \infty$—whose existence and continuity is elementary since all functions $y(x)$ are solutions of (1)—is a bounded function and likewise a solution of (1). Since $y(x)$ is concave and increasing in (a, b), we have

$$\int_a^b y'^2 dx \leqq y'(a) \int_a^b y' dx = y'(a)y(b) \leqq my'(a).$$

Since $\lambda(a)$ was shown to be positive, $\int_a^b y'^2 dx$ is bounded from below by a positive number, and it follows that, for $b \rightarrow \infty$, the values of $y'(a)$ have a positive lower bound. If $y_0(x)$ is one of the limit functions just mentioned, we have thus shown that $y_0'(a) > 0$ and that, therefore, $y_0(x)$ is not identically zero. Since $y_0(x)$, as the limit of non-negative functions, cannot take negative values in (a, ∞), this completes the proof of Theorem IX.

It may be noted that Theorem IX is sharp in the sense that (40) cannot be replaced by

$$\int^\infty x^{1-\delta} F(\beta x, x)dx < \infty$$

where δ is an arbitrarily small positive number. This condition is satisfied by equation (5) which was shown to have no properly nonoscillatory solutions.

A necessary condition for the existence of properly nonoscillatory solutions can be obtained by combining Theorems VII and VIII, although there remains the difficulty of ascertaining whether or not a given equation (1) has the uniqueness property required in Theorem VII. As the following result shows, this difficulty can be avoided if additional information is available regarding the growth of the solution in question.

THEOREM X. *If* (1) *has a properly nonoscillatory solution* $u(x)$ *for which*

$$x^{-1/2}u(x) > c > 0 \tag{42}$$

for large x, *then*

$$x\int_x^\infty F(mx, x)dx < M, \qquad x > 0,$$

for suitable positive constants m and M.

In view of Theorem VIII, Theorem X will be proved if it is shown that condition (42) guarantees that $\lambda(a) > 0$ for some a. If it were true that $\lambda(a) = 0$, it would follow that the functions $y(x)$ of Theorem IV tend to the solution $y(x) \equiv 0$ for $b \to \infty$ and that, therefore, $0 < y'(a) < u'(a)$ for sufficiently large b. It is easy to show that, in this case, the curves $\zeta = y(x)$ and $\zeta = u(x)$ must intersect in (a, b). Indeed, if there were no intersection we would have $y(x) < u(x)$ in (a, b) and thus, by (1),

$$\begin{aligned} u'(b)y(b) &= u'(b)y(b) - u(b)y'(b) \\ &= \int_a^b uy[F(y^2, x) - F(u^2, x)]dx < 0, \end{aligned}$$

which is absurd. In view of $y(x) \leqq y'(a)(x-a)$, these points of intersection, say $x = x_0$, must tend to infinity as $b \to \infty$. But, for sufficiently large x_0,

$$0 < c^2x_0 \leqq u^2(x_0) = y^2(x_0) \leqq (x_0 - a)\int_a^b y'^2 dx \tag{43}$$

and, in view of (22), this implies a contradiction to the assumption $\lambda(a) = 0$.

The following result, established by a similar argument, shows that any equation (1) which has properly nonoscillatory solutions must also have properly nonoscillatory solutions for which lim inf $x^{-1/2}u(x)$ for $x \to \infty$ is bounded.

THEOREM XI. *If equation* (1) *has a properly nonoscillatory solution whose last zero occurs at* $x = a$, *then it must also have a solution* $u(x)$ *of the same type for which*

$$\liminf_{x\to\infty} x^{-1/2}u(x) < M < \infty. \tag{44}$$

If $y(x)$ has the same meaning as in the proof of Theorem X, there are two possibilities: the curves $\zeta = y(x)$ may intersect the curve $\zeta = u(x)$ in (a, b) for all sufficiently large b, or else there exists a sequence b_ν with $b_\nu \to \infty$ such that the corresponding curves $\zeta = y(x)$ do not intersect $\zeta = u(x)$. In the second case it follows from the preceding argument that $u(x) \leqq y(x)$ in (a, b). Hence $u^2(x) \leqq y^2(x) \leqq (x-a)\int_a^b y'^2 dx$, and an application of the corollary to Theorem

V (formula (31)) establishes (44). In the first case there are again two possibilities, according as the points of intersection do or do not tend to infinity as $b\to\infty$. If they do, we use the same argument as in the proof of Theorem X, and (44) follows by letting $x_0\to\infty$ in (43). Finally, if x_0 tends to a finite value x_1 as $b\to\infty$, we observe that, because of the concavity of the curve $\zeta=y(x)$, we have $y'(a)\geqq y(x_0)(x_0-a)=u(x_0)(x_0-a)$. Hence, $y'(a)$ has a positive lower bound as $b\to\infty$ and there exists a subsequence of functions $y(x)$ which approaches a nontrivial solution $v(x)$ of (1). For elementary reasons $y(x)$ converges uniformly to $v(x)$ in any finite interval. Since $y^2(x)\leqq M(x-a)$ in (a, b), we have $v^2(x)\leqq M(x-a)$ in (a, ∞), and the proof is complete.

Theorem XI can be used to obtain additional information concerning the character of the properly nonoscillatory solutions of an equation (1) for which (11) holds. It was shown in the proof of Theorem II that, for any $a>0$, an equation of this type has properly nonoscillatory solutions which vanish at $x=a$ and are positive and $O(x)$ in (a, ∞). In view of Theorems II and XI, we therefore have the following result.

An equation (1) *which satisfies the hypotheses of Theorem* II *has, for every positive a, a properly nonoscillatory and bounded solution whose last zero is at $x=a$.*

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